16.2 Line Integrals

Integration along a curve $\mathcal{C}$.

Consider a curve $\mathcal{C}$ in $\mathbb{R}^2$

$$x = x(t), \quad y = y(t), \quad a \le t \le b.$$

or equivalently

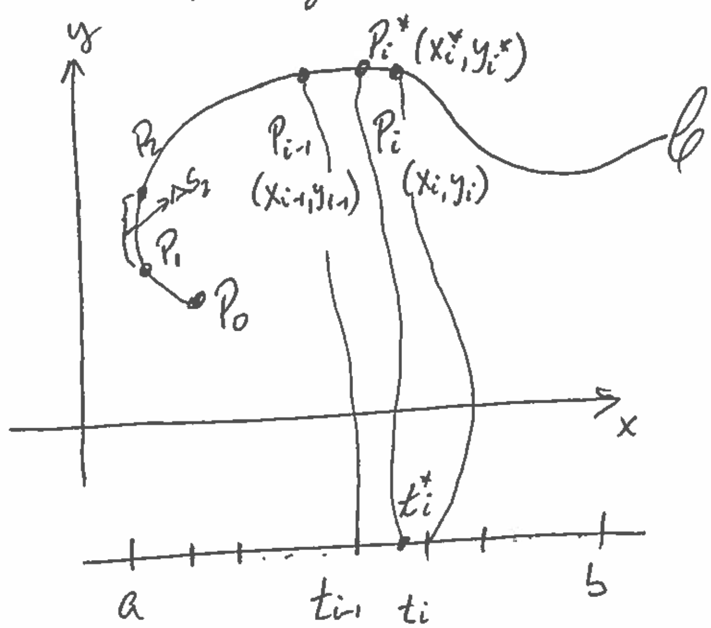
$$\mathcal{C}: \vec{r}(t) = \langle x(t), y(t) \rangle$$

We assume $\mathcal{C}$ is smooth, which means

$\vec{r}\,'(t)$ is conts. and $\vec{r}\,'(t) \neq \vec{0}$.

We divide the interval of t $[a,b]$ into n

Subintervals of equal width



$x_i = x(t_i)$

$y_i = y(t_i)$

The corresp. points $P_i(x_i, y_i)$ divide $\mathcal{C}$ into n subarcs with lengths $\Delta s_1, .., \Delta s_i, .., \Delta s_n$

P_i^* is in ith subarcs

and corresponds to $t_i^* \in [t_{i-1}, t_i]$.

Now, if $f(x,y)$ is a function of two variables whose domain includes $\mathcal{C}$, we form the analogous of a Riemann Sum

$$\sum_{i=1}^{n} f(x_i^*, y_i^*) \Delta S_i$$

Then, we consider the

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*) \Delta S_i \qquad (2.1)$$

if this limit exists, we use the notation

$$\int_{\mathcal{C}} f(x,y)\, ds \qquad (2.2)$$

and this limit is called

Line Integral of f along $\mathcal{C}$.

Since,

$$S(t) = \int_a^t \sqrt{x'^2(t) + y'^2(t)}\, dt$$

length of $\mathcal{C}$ between a and t.

$$\Rightarrow \frac{ds}{dt} = \sqrt{x'^2(t) + y'^2(t)} \quad (3.1)$$

Then, we have the following result

If $f(x,y)$ is a conts. function, then the limit (2.1) always exists and we have the formula:

$$\int_{\mathcal{C}} f(x,y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{x'^2(t) + y'^2(t)}\, dt$$

The value of the integral does not depend on the parametization of the curve.

Notice, from (3.1)

$$ds = |\vec{r}'(t)|\, dt = \sqrt{x'^2(t) + y'^2(t)}\, dt$$

Add Line integrals in space.

16.2 #12)

$$I = \int_{\mathcal{C}} (2x + 9z)\,ds, \qquad \mathcal{C}: x(t) = t,\ y(t) = t^2,\ z(t) = t^3$$
$$0 \le t \le 1.$$

$$I = \int_{t=0}^{t=1} \left[2t + 9t^3\right] \sqrt{(1)^2 + (2t)^2 + (3t^2)^2}\,dt$$

$$= \int_0^1 \left[2t + 9t^3\right] \sqrt{1 + 4t^2 + 9t^4}\,dt$$

If $u = 1 + 4t^2 + 9t^4 \Rightarrow du = (8t + 36t^3)\,dt$

or $du = 4\,(2t + 9t^3)\,dt$

Thus,
$$I = \frac{1}{4}\int_{u=1}^{u=14} u^{1/2}\,du = \frac{1}{4}\,\frac{u^{3/2}}{3/2}\Bigg|_1^{14} = \frac{1}{6}\left[(14)^{3/2} - 1\right]$$

16.2 # 9) (Discuss this in class if time allows)

$$I = \int_{\mathcal{C}} xyz\, ds, \qquad \mathcal{C}: \begin{cases} x(t) = 2\sin t \\ y(t) = t \\ z(t) = -2\cos t \end{cases} \quad 0 \le t \le \pi$$

$$I = -\int_0^{\pi} 4t \sin t \cos t \sqrt{4\cos^2 t + 1^2 + 4\sin^2 t}\, dt$$

$$= -4\int_0^{\pi} t \sin t \cos t \sqrt{5}\, dt = -\cancel{4}^{\,2}\,\frac{\sqrt{5}}{\cancel{2}_{\,1}} \int_0^{\pi} t\big(\overbrace{2\sin t \cos t}^{=\sin(2t)}\big)\, dt$$

Now,

$$\int \underbrace{t}_{f}\, \underbrace{\sin(2t)}_{g'}\, dt \overset{\text{I.P.P.}}{=} \underbrace{-t\,\frac{\cos(2t)}{2} + \int \frac{\cos(2t)}{2}\, dt}_{fg - \int f'g}$$

Thus,

$$I = -2\sqrt{5}\left[\left(-t\,\frac{\cos(2t)}{2}\right)\Bigg|_0^{\pi} + \frac{1}{2}\,\frac{\sin(2t)}{2}\Bigg|_0^{\pi}\right] =$$

$$= -2\sqrt{5}\left[-\frac{\pi}{2} + \frac{1}{4}\cdot 0\right] = \sqrt{5}\,\pi$$

If $\mathcal{C}$ is piecewise-smooth. It means

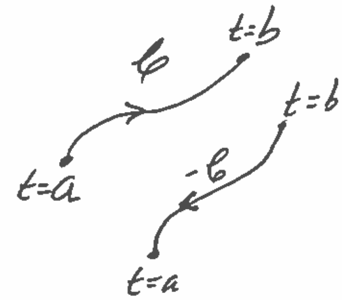
$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_n$ and init. point of $\mathcal{C}_{i+1}$ is final point of $\mathcal{C}_i$

and each $\mathcal{C}_i$ is smooth, then

$$\int_{\mathcal{C}} f(x,y,z)\, ds = \int_{\mathcal{C}_1} f\, ds + \cdots + \int_{\mathcal{C}_n} f\, ds.$$

<u>PROPERTIES:</u>

$$\int_{-\mathcal{C}} f(x,y)\, ds = \int_{\mathcal{C}} f(x,y)\, ds$$



<u>Line Integrals of $f(x,y,z)$ with respect to x, y, and z.</u>

<u>Definition</u>

$$\int_{\mathcal{C}} f(x,y,z)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\, \Delta x_i$$

partition along x-axis.

Analogously,

$$\int_{\mathcal{C}} f(x,y,z)\, dy = \lim_{n\to\infty} \sum f(*)\, \Delta y_i$$

$$\int_{\mathcal{C}} f(x,y,z)\, dz = \lim_{n\to\infty} \sum f(*)\, \Delta z_i$$

Using a parametric representation of $\mathcal{C}$: $\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \\ \quad a \le t \le b \end{cases}$

$$\int_{\mathcal{C}} f(x,y,z)\,dx = \int_a^b f(x(t), y(t), z(t))\, x'(t)\,dt$$

$$\int_{\mathcal{C}} f\,dy = \int_a^b f(\ )\, y'(t)\,dt$$

$$\int_{\mathcal{C}} f\,dz = \int_a^b f(\ )\, z'(t)\,dt$$

A combined integration w.r.t. x and y or w.r.t. x, y, and z frequently appears.

It means

$$I = \int_{\mathcal{C}} P(x,y)\,dx + \int_{\mathcal{C}} Q(x,y)\,dy = \int_{\mathcal{C}} P(x,y)\,dx + Q(x,y)\,dy$$

Discuss example in book (Example 4). Pow. Point slides.

Notice

$$\int_{-\mathcal{C}} f(x,y)\,dx = -\int_{\mathcal{C}} f(x,y)\,dx$$

This is because $\vec{x}'(t)$ change sign.

Line Integrals of Vector fields.

In 1-d

If $f(x)$ is force acting on particle moving from $x=a$ to $x=b$.

$$\text{Work} = \int_a^b f(x)\,dx.$$

We also know,

if $\vec{F}$ (const) is a force in 3-d, then the work done to move an object from a point P to Q. is given by

$$W = \vec{F}\cdot\vec{PQ} = \vec{F}\cdot\vec{D}.$$

If we have a force given by the vector field

$$\vec{F}(x,y,z) = P(x,y,z)\,\hat{i} + Q(x,y,z)\,\hat{j} + R(x,y,z)\,\hat{j}$$

Continuous force field in $\mathbb{R}^3$.

Want to compute work done by this force $\vec{F}$ moving a particle along a smooth curve $\mathcal{C}$.

Show P.P. graph with partition on $\mathcal{C}$.

$$\text{Work} = W = \lim_{n\to\infty} \sum_{i=1}^{n} \left[F(x_i^*, y_i^*, z_i^*) \cdot T(x_i^*, y_i^*, z_i^*)\,\Delta S_i \right.$$

If this limit exists

$$\overset{\text{def}}{=} \int_{\mathcal{C}} \vec{F}(x,y,z) \cdot \vec{T}(x,y,z)\,ds = \int_{\mathcal{C}} \vec{F} \cdot \vec{T}\,ds.$$

If $\mathcal{C}$: $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, $a \le t \le b$

$$\Rightarrow \vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} \quad \text{and}$$

$$W = \int_a^b F(\vec{r}(t)) \cdot \frac{\vec{r}\,'(t)}{\cancel{|\vec{r}\,'(t)|}}\,\cancel{|\vec{r}\,'(t)|}\,dt = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

Definition

$\vec{F}$ conts vector field defined on a Smooth Curve $\mathscr{C}$ given by $\vec{r}(t)$, $a \le t \le b$

Then "Line Integral of $\vec{F}$ along $\mathscr{C}$"

$$\int_{\mathscr{C}} \vec{F} \cdot \hat{T} \, ds$$

Example

Alternative Vector Fields:

$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = \int_{\mathcal{C}} F(x,y,z)\cdot \vec{T}(x,y,z)\, ds =$$

↳ unit tang. vector of $\mathcal{C}$

$$= \int_{\mathcal{C}} \vec{F}(\vec{r}(t))\cdot \vec{r}'(t)\, dt$$

If $\vec{F}(x,y,z) = P(x,y,z)\,\hat{\imath} + Q(x,y,z)\,\hat{\jmath} + R(x,y,z)\,\hat{k}$

and $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

Then, $\int_{\mathcal{C}} \vec{F}(r(t))\cdot \vec{r}'(t)\, dt =$

$$= \int_a^b \left[P(x(t),y(t),z(t))\, x'(t) + Q(x(t),y(t),z(t))\, y'(t) + R(x(t),y(t),z(t))\, z'(t) \right] dt$$

$$= \int_a^b P(x(t),y(t),z(t))\, x'(t)\, dt + \cdots + \int_a^b R(x(t),y(t),z(t))\, z'(t)\, dt =$$

$$= \int_{\mathcal{C}} P(x,y,z)\, dx + \cdots + \int_{\mathcal{C}} R(x,y,z)\, dz .$$